A PROBLEM ON EDGE-MAGIC LABELINGS OF CYCLES

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Abstract. Kotzig and Rosa defined in 1970 the concept of edge-magic labelings as follows: let G be a simple (p,q)-graph (that is, a graph of order p and size q without loops or multiple edges). A bijective function $f: V(G)\cup E(G) \to \{1,2,\ldots,p+q\}$ is an edge-magic labeling of G if $f(u)+f(uv)+f(v)=k$, for all $uv \in E(G)$. A graph that admits an edge-magic labeling is called an edge-magic graph, and k is called the magic sum of the labeling. An old conjecture of Godbold and Slater sets that all possible theoretical magic sums are attained for each cycle of order $n \geq 7$. Motivated by this conjecture, we prove that for all $n_0 \in \mathbb{N}$, there exists $n \in \mathbb{N}$, such that the cycle C_n admits at least n_0 edge-magic labelings with at least n_0 mutually distinct magic sums. We do this by providing a lower bound for the number of magic sums of the cycle C_n, depending on the sum of the exponents of the odd primes appearing in the prime factorization of n.

1. Introduction

For the graph theory terminology and notation not defined in this paper we refer the reader to either one of the following sources [3, 5, 8, 15]. Kotzig and Rosa [10] defined in 1970 the concept of edge-magic labelings as follows: let G be a simple (p,q)-graph (that is, a graph of order p and size q without loops or multiple edges). A bijective function $f: V(G)\cup E(G) \to \{1,2,\ldots,p+q\}$ is an *edge-magic labeling* of G if $f(u)+f(uv)+f(v)=k$, for all $uv \in E(G)$. A graph that admits an edge-magic labeling is called an *edge-magic graph*, and k is called the *valence*, the *magic sum* [15] or the *magic weight* [3] of the labeling.

Godbold and Slater introduced in [9] the following conjecture.

Conjecture 1.1. [9]. *For $n = 2t+1 \geq 7$ and $5t+4 \leq j \leq 7t+5$ there is an edge-magic labeling of C_n with magic sum $k = j$. For $n = 2t \geq 4$ and $5t+2 \leq j \leq 7t+1$ there is an edge-magic labeling of C_n with magic sum $k = j$.*

We mention that, the lower bound (respectively the upper bound) on the magic sum comes from assigning the lowest (respectively the highest) numbers to the vertices of the cycle. Motivated by this conjecture we introduce the following theorem. The goal of this paper is to prove it.

Theorem 1.2. *For all $n_0 \in \mathbb{N}$, there exists $n \in \mathbb{N}$, such that the cycle C_n admits at least n_0 edge-magic labelings with at least n_0 mutually distinct magic sums.*

2. The tools

Figueroa-Centeno et al. introduced in [7] the following definition: let D be a digraph and let $\Gamma = \{F_i\}_{i=1}^m$ be a family of digraphs such that $V(F_i) = V$ for every

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$i \in \{1, 2, \ldots, m\}$. Consider a function $h : E(D) \longrightarrow \Gamma$. Then the product $D \otimes_h \Gamma$ is the digraph with vertex set $V(D) \times V$ and $((a, b), (c, d)) \in E(D \otimes_h \Gamma)$ if and only if $(a, c) \in E(D)$ and $(b, d) \in E(h(a, c))$. The adjacency matrix of $D \otimes_h \Gamma$, namely $A(D \otimes_h \Gamma)$, is obtained by replacing every 0 entry of $A(D)$, the adjacency matrix of D, by the $|V| \times |V|$ null matrix and every 1 entry of $A(D)$ by $A(h(a, c))$.

The following restriction of edge-magic labelings introduced independently by Acharya and Hegde [1] and by Enomoto et al. [6] will prove to be of great help in the rest of this document. Let G be a (p, q)-graph. Then G is a *super edge-magic* graph [1, 6] if there is an edge-magic labeling of G, namely $f : V(G) \cup E(G) \to \{i\}_{i=1}^{p+q}$, with the extra property that $f(V(G)) = \{i\}_{i=1}^{p}$. The labeling f is called a *super edge-magic labeling* of G. All cycles are edge-magic [9]. However, a cycle C_p is super edge-magic if and only if p is odd [6]. As in [7], a digraph D is said to admit a labeling l if its underlying graph, und(D), admits l. From now on, let $\mathcal{S}_p$ be the set of all 1-regular super edge-magic labeled digraphs of odd order p, $p \geq 3$, where each vertex takes the name of the label assigned to it. Then we have the following theorem.

Theorem 2.1. [7] *Let D be a (super) edge-magic digraph, and let $h : E(D) \to \mathcal{S}_p$ be any function. Then und$(D \otimes_h \mathcal{S}_p)$ is (super) edge-magic.*

The key point in the proof (see also [11]) is to rename the vertices of D and each element of $\mathcal{S}_p$ after the labels of their corresponding (super) edge-magic labeling f and their super edge-magic labelings respectively and define the labels of the product as follows: (i) the vertex $(i, j) \in V(D \otimes_h \mathcal{S}_p)$ receives the label: $p(i-1)+j$ and (ii) the arc $((i, j), (i', j')) \in E(D \otimes_h \mathcal{S}_p)$ receives the label: $p(e-1) + (3p + 3)/2 - (j + j')$, where e is the label of (i, i') in D. Thus, for each arc $((i, j), (i', j')) \in E(D \otimes_h \mathcal{S}_p)$, coming from an arc $e = (i, i') \in E(D)$ and an arc $(j, j') \in E(h(i, i'))$, the sum of labels is constant and equal to: $p(i + i' + e - 3) + (3p + 3)/2$. That is, $p(\sigma_f - 3) + (3p + 3)/2$, where σ_f denotes the magic sum of the labeling f of D. Therefore, we obtain the following proposition.

Proposition 2.2. *Let $\check{f}$ be the edge-magic labeling of the graph und$(D \otimes_h \mathcal{S}_p)$ obtained in Theorem 2.1 from a labeling f of D. Then the magic sum of $\check{f}$, $\sigma_{\check{f}}$, is given by the formula*

$$\sigma_{\check{f}} = p(\sigma_f - 3) + \frac{3p + 3}{2}, \tag{2.1}$$

where σ_f is the magic sum of f.

Corollary 2.3. *Let D be an edge-magic digraph and assume that there exist two edge-magic labelings of D, f and g, such that $\sigma_f \neq \sigma_g$. If we denote by $\check{f}$ and $\check{g}$ the edge-magic labelings of the graph und$(D \otimes_h \mathcal{S}_p)$ when using the edge-magic labelings f and g of D respectively, then we get*

$$|\sigma_{\check{f}} - \sigma_{\check{g}}| \geq 3.$$

Proof. Since $\sigma_f \neq \sigma_g$, we get the inequality $|\sigma_f - \sigma_g| \geq 1$. Thus, by using (2.1), we obtain that $|\sigma_{\check{f}} - \sigma_{\check{g}}| = |p(\sigma_f - \sigma_g)| \geq 3$. □

The following two results appear in [15].

Theorem 2.4. [15] *Every odd cycle C_n has an edge-magic labeling with magic sum $3n + 1$ and an edge-magic labeling with magic sum $3n + 2$.*

Theorem 2.5. [15] *Every even cycle C_n has an edge-magic labeling with magic sum $(5n+4)/2$.*

Next, we state the following two structural results. We denote by $\overrightarrow{C_n}$ and by $\overleftarrow{C_n}$ the two possible strong orientations of the cycle C_n, where the vertices of C_n are the elements of the set $\{i\}_{i=1}^n$.

Theorem 2.6. [7] *Let $h : E(\overrightarrow{C_m}) \longrightarrow \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$ be any constant function. Then und$(\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\}) = \mathrm{gcd(m,n)C_{lcm[m,n]}}$.*

Theorem 2.7. [2] *Let $m, n \in \mathbb{N}$ and consider the product $\overrightarrow{C}_m \otimes_h \{\overrightarrow{C}_n, \overleftarrow{C}_n\}$ where $h : E(\overrightarrow{C}_m) \longrightarrow \{\overrightarrow{C}_n, \overleftarrow{C}_n\}$. Let g be a generator of a cyclic subgroup of $\mathbb{Z}_n$, namely $\langle g\rangle$, such that $|\langle g\rangle| = k$. Also let $N_g(h^-) < m$ be a natural number that satisfies the congruence relation $m - 2N_g(h^-) \equiv g \ (mod\ \ n)$.*

If the function h assigns $\overleftarrow{C}_n$ to exactly $N_g(h^-)$ arcs of $\overrightarrow{C}_m$ then the product

$$\overrightarrow{C}_m \otimes_h \{\overrightarrow{C}_n, \overleftarrow{C}_n\}$$

consists of exactly n/k disjoint copies of a strongly oriented cycle $\overrightarrow{C}_{mk}$. In particular if gcd$(g, n) = 1$, then $\langle g\rangle = \mathbb{Z}_n$ and if the function h assigns $\overleftarrow{C}_n$ to exactly $N_g(h^-)$ arcs of $\overrightarrow{C}_m$ then

$$\overrightarrow{C}_m \otimes_h \{\overrightarrow{C}_n, \overleftarrow{C}_n\} \cong \overrightarrow{C}_{mn}.$$

Corollary 2.8. *Let $n \geq 3$ be an odd integer and suppose that $m \geq 3$ is an integer such that either m is odd or $m \geq n$. Then there exists a function $h : E(\overrightarrow{C_m}) \to \{\overrightarrow{C_n}, \overleftarrow{C_n}\}$ such that*

$$\overrightarrow{C_m} \otimes_h \{\overrightarrow{C_n}, \overleftarrow{C_n}\} \cong \overrightarrow{C_{mn}}.$$

Proof. We have that $\langle 1\rangle = \mathbb{Z}_n$ and, since n is odd, the congruence relation $m - 2r \equiv 1 \ (mod\ \ n)$ can be solved, with $0 < r < m$. Therefore, inheriting the notation of Theorem 2.7, by considering any function h with $N_1(h^-) = r$, we get the desired result. □

3. Proof of the main result

We start this section by showing four edge-magic labelings of C_3 with consecutive magic sums in Fig. 1.

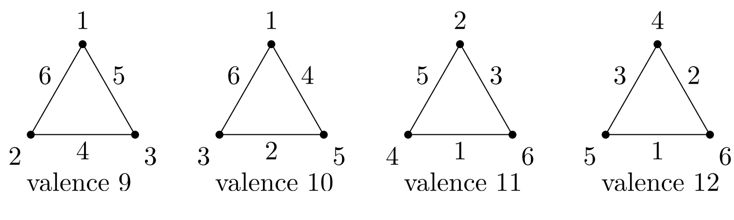

Figure 1. Edge-magic labelings of C_3.

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We already know that C_3 admits 4 edge-magic labelings with 4 consecutive edge-magic magic sums (notice that the labeling corresponding to magic sum 9 is super edge-magic). Call these labelings l_1, l_2, l_3, l_4, where the magic sum of l_i is less than the magic sum of l_j if and only if $i < j$ ($i, j \in \{1, 2, 3, 4\}$), and denote by $C_3^{l_i}$ the copy of C_3, where each vertex takes the name of the label that l_i has assigned to it. Also let $\overrightarrow{C_3^{l_i}}$ be the digraph obtained from $C_3^{l_i}$ with the edges oriented cyclically. Recall that, we denote by $\overrightarrow{C_3}$ and $\overleftarrow{C_3}$ the two possible strong orientations of C_3, where the vertices of C_3 are labeled in a super edge-magic way. Let $\Gamma = \{\overrightarrow{C_3}, \overleftarrow{C_3}\}$.

By Corollary 2.8, for all $i \in \{1, 2, 3, 4\}$ there exists a function $h_i : E(\overrightarrow{C_3^{l_i}}) \to \Gamma$ such that $\text{und}(\overrightarrow{C_3^{l_i}} \otimes_{h_i} \Gamma) \cong C_9$. Also, any two magic sums of the labelings obtained for $\overrightarrow{C_3^{l_i}} \otimes_{h_i} \Gamma$ differ, by Corollary 2.3, by at least three units. But we know by Theorem 2.4 that magic sums 28 and 29 appear for different edge-magic labelings of C_9. Hence, the cycle C_9 admits at least 5 edge-magic labelings with 5 different magic sums. Let the labelings that provide these magic sums be l_i^1, where the magic sum of l_i^1 is less than the magic sum of l_j^1 if and only if $i < j$ ($i, j \in \{1, 2, \ldots, 5\}$).

If we repeat the process with $\overrightarrow{C_9^{l_i^1}} \otimes_{h_i^1} \Gamma$, where $h_i^1 : E(\overrightarrow{C_9^{l_i^1}}) \to \Gamma$ is a function as in Corollary 2.8, we obtain 5 edge-magic labelings of C_{27} with 5 different magic sums. But, again by Corollary 2.3, either magic sum 82 or magic sum 83, does not appear among these 5 magic sums, since among these 5 magic sums no two magic sums are consecutive. But we know by Theorem 2.4 that these two magic sums, 82 and 83, appear for an edge-magic labeling of C_{27}. Hence, there are at least 6 magic sums for edge-magic labelings of C_{27}.

Repeating this process inductively, we obtain that each cycle of order 3^α admits at least $3 + \alpha$ edge-magic labelings with at least $3 + \alpha$ mutually different magic sums. Therefore, we get the desired result. $\square$

Notice that, using a similar idea to the one in the proof of Theorem 1.2, we can obtain the next theorem.

Theorem 3.1. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the unique prime factorization (up to ordering) of an odd number n. Then C_n admits at least $1 + \sum_{i=1}^k \alpha_i$ edge-magic labelings with at least $1 + \sum_{i=1}^k \alpha_i$ mutually different magic sums.*

Using Theorem 2.5 and the previous construction, we can prove the next theorem.

Theorem 3.2. *Let $n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the unique prime factorization of an even number n, with $p_1 > p_2 > \ldots > p_k$. Then C_n admits at least $\sum_{i=1}^k \alpha_i$ edge-magic labelings with at least $\sum_{i=1}^k \alpha_i$ mutually different magic sums. If $\alpha \geq 2$ this lower bound can be improved to $1 + \sum_{i=1}^k \alpha_i$.*

Proof. Assume first that $\alpha \geq 2$. By Theorem 2.5, the cycle of order 2^α has an edge-magic labeling l with magic sum $5 \cdot 2^{\alpha-1} + 2$. Let $C_{2^\alpha}^l$ be the copy of C_{2^α}, where each vertex takes the name of the label that l has assigned to it and let $\Gamma_i = \{\overrightarrow{C_{p_i}}, \overleftarrow{C_{p_i}}\}$, where the vertices of C_{p_i} are labeled in a super edge-magic way, for each $i = 1, 2, \ldots, k$. Also let $\overrightarrow{C_{2^\alpha}^l}$ be the digraph obtained from $C_{2^\alpha}^l$ such that the edges have been oriented cyclically.

By Theorem 2.6, any constant function $h : E(\overrightarrow{C^l_{2^\alpha}}) \to \Gamma_1$ gives $C_{2^\alpha \cdot p_1} \cong \mathrm{und}(\overrightarrow{C^l_{2^\alpha}} \otimes_h \Gamma_1)$. Notice that, by Proposition 2.2, the induced edge-magic labeling on $C_{2^\alpha \cdot p_1}$ has magic sum:

$$p_1(\sigma_l - 3) + \frac{3p_1 + 3}{2} = 5p_1 \cdot 2^{\alpha - 1} + \frac{p_1 + 3}{2}.$$

Since by Theorem 2.5, the cycle $C_{2^\alpha \cdot p_1}$ has an edge-magic labeling with magic sum $5p_1 \cdot 2^{\alpha-1} + 2$, we get that $C_{2^\alpha \cdot p_1}$ admits 2 edge-magic labelings with 2 different magic sums. Assume that $\sum_{i=1}^k \alpha_i \geq 2$, otherwise the result is proved, and call these labelings l_1, l_2, where the magic sum of l_1 is less than the magic sum of l_2. Denote by $C^{l_i}_{2^\alpha \cdot p_1}$ the copy of $C_{2^\alpha \cdot p_1}$, where each vertex takes the name of the label that l_i has assigned to it. Also let $\overrightarrow{C^{l_i}_{2^\alpha \cdot p_1}}$ be the digraph obtained from $C^{l_i}_{2^\alpha \cdot p_1}$ such that the edges have been oriented cyclically.

By Corollary 2.8, for each $i \in \{1, 2\}$ and for some fixed $j \in \{1, 2, \ldots, k\}$, there exists a function $h_i : E(\overrightarrow{C^{l_i}_{2^\alpha \cdot p_1}}) \to \Gamma_j$ such that $\mathrm{und}(\overrightarrow{C^{l_i}_{2^\alpha \cdot p_1}} \otimes_{h_i} \Gamma_j) \cong C_{2^\alpha \cdot p_1 p_j}$. We take $j = 1$ when $\alpha_1 > 2$, and $j = 2$ when $\alpha_1 = 1$. Also, the two magic sums of the labelings obtained from $\overrightarrow{C}^{l_i}_{2^\alpha \cdot p_1} \otimes_{h_i} \Gamma_j$ differ, by Corollary 2.3, by at least three units. Moreover, the minimum of them, that is $p_j(\sigma_{l_1} - 3) + (3p_j + 3)/2 = 5p_1 p_j \cdot 2^{\alpha-1} + (p_j + 3)/2$, is bigger than the magic sum guaranteed by Theorem 2.5. Hence, the cycle $C_{2^\alpha \cdot p_1 p_j}$ has at least three edge-magic labelings with at least three mutually different magic sums.

Repeating this process inductively, following the order of primes, we obtain that each cycle of order $2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ admits at least $1 + \sum_{i=1}^k \alpha_i$ edge-magic labelings with at least $1 + \sum_{i=1}^k \alpha_i$ mutually different magic sums.

Assume now that $\alpha = 1$. In this case, we proceed as in the case $\alpha \geq 2$, but starting with the cycle of length $2^\alpha p_1$. Therefore, we get the desired result. □

3.1. **Conclusions.** Renewed interest seems to be growing up lately on the study of the magic sums of edge-magic and super edge-magic labelings (see [12, 13] for instance). In this paper we have concentrated our efforts in the study of the set of edge-magic magic sums for cycles. This is an old problem that appeared in [9] and that has remained unsolved for 15 years. Very little progress has been made towards a solution of it since then. In fact, for many years only four magic sums have been known for C_n, except for small values of n, where the problem has been treated using computers (see [4]). It has not been until 2009 that a paper has appeared [14], in which the author has proved a result similar to the one introduced in this paper. However the method used in [14] and the method introduced in this paper are absolutely different. We feel that to try to combine both methods can be a very interesting line of research for the future. So far, we remark that concerning this problem about the valences of C_n we have only two different methods that allow us to show that the number of magic sums of the cycle C_n grows unbounded for the values of n. However the original question found in [9] remains unsolved and we feel that, at this point, we are very far away from a final solution.

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